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Giving up logical atomism? Some remarks on Wittgenstein's *Some Remarks on Logical Form* (1929)**

ABSTRACT

Elementary (atomic) and complex structures can be characterized with respect to its composition as well as to its decomposition. (a) The rules to (de-)compose *elementary* structures are non-recursive. (b) The rules to (de-)compose *complex* structures are usually conditioned and recursive. Using this differentiation four aspects of logical atomism with respect to Wittgenstein's *Tractatus* are discussed: (atomic) facts, elementary propositions, objects/names and substitution rule. With respect to these aspects we investigate Wittgenstein's modifications of his position towards logical atomism in *Some Remarks on Logical Form*. Against Wittgenstein's position the consequences of using truth-functions within atomic propositions are checked. Finally we discuss analogies between the exclusion of atomic propositions on colour and the exclusion of atomic propositions describing positions of chess pieces on a board. The mutual exclusion of descriptions is related to the structure of the relevant logical space relative to the basic rules of the game. These considerations indicate one possible path to the concept of *language games* in Wittgenstein's later philosophy.

Keywords: Logical Atomism; Recursiviness; Truth Functionality; Middle Wittgenstein; *Tractatus*.

RESUMO

Estruturas elementares (atômicas) e complexas podem ser caracterizadas tanto em relação a sua composição quanto em relação a sua decomposição. (a) as regras para (de-)compor estruturas *elementares* não são recursivas. (b) As regras para (de-)compor estruturas *complexas* são comumente condicionadas e recursivas. Ao usar esta diferenciação quatro aspectos do atomismo lógico do *Tractatus* de Wittgenstein são discutidos: fatos (atômicos), proposições elementares, objetos/nomes e regras de substituição. Considerando estes aspectos nós investigamos as modificações de sua posição em relação ao atomismo lógico em *Some Remarks on Logical Form*. Contra a posição de Wittgenstein as consequencias de se usar funções de verdade dentro de proposições atômicas são analisadas. Finalmente, nós discutimos analogias entre a exclusão de proposições atômicas descrevendo posições das pecas de xadrez em um tabuleiro. A exclusão mútua de descrições é relacionada à estrutura do espaço lógico relativo às regras básicas do jogo. Estas considerações indicam um possível caminho para o conceito de *jogos de linguagem* na Filosofia tardia de Wittgenstein.

Palavras-chave: Atomismo lógico; Recursividade; Vero-funcionalidade; Wittgenstein intermediário; *Tractatus*.

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** In the Text I use abbreviations for some of Wittgenstein's writings: T for Tractatus logico-philosophicus, RLF for Some Remarks on Logical Form, PR for Philosophical Remarks and RFM for Remarks on the Foundation of Mathematics.

1 Aspects of Logical Atomism in the *Tractatus*

There are two ways of speaking about the totality of facts: *composition* ("buttom-up") and *decomposition* ("top-down"). Both ways are indistinguishable from a holistic point of view. The totality of facts is simply given. *Composition* means that we start with some input which is simple in some sense and end up in a complex structure which is the (recursively "created") "product" of uniform procedures. Decomposition means that we start with any complex structure and divide this structure into "well-formed" substructures of the same kind (same logical type). This procedure stops at simple structures which cannot be decomposed further in the same manner or cannot be decomposed further at all. In the first case we can speak of *elementary* structures in the latter case of *atomic* objects.

Let us have a look at a very familiar case of composition and decomposition in logical syntax: the case of composing and decomposing (well-formed) formulas. Usually we use formation rules to do this. But we have two different kinds of rules: (I) *unconditioned* rules and (II) *conditioned* rules. An unconditioned rule in classical propositional logic can be formulated this way: (R1) A propositional variable (of the form p_i) standing alone is a formula. From this propositional point of view we can look at some formula *p* as a *real atom* if we have no further rule telling us something about the substructure of *p*. But from the point of first-order logic we can characterize *elementary* formulas using the *conditioned* rule (R2): *If* $i_1,...,i_n$ are individual constants (objects, things, names) or individual variables (variables of objects, things. names) and R^n any *n*-ary predicate *then* the concatenation $R^n\, i_1,...,i_n$ is a formula. Because this formulation already seems to presuppose a function-argument-structure – predicates of the form R^n are *n*-ary functions and individual terms of the form i_j $(1 \leq j \leq n)$ are arguments of such functions – we can give this rule a more neutral formulation: (R2*) *If* $i_1,...,i_n$ are objects of any kind then i_1-...- i_n is a formula. That means we can represent the formula "*aRb*" (T3.1432) of the form using the more neutral form "$a - b$". Please note that the rules (R2) and (R2*) are conditioned but *not recursive* rules. These rules cannot be applied to itself! Finally we have a conditioned rule which allows to create *complex* formulas: (R3) *If* $R^n A_1,...,A_n$ are formulas (already created by any of the admitted rules of our logic including this rule itself) and φ_i^n is any *n*-ary classical operator with $n \geq 1$ and $1 \leq i \leq 2^{2^n}$ (associated with an extensional and 2-valued truth-function) then $\varphi_i^n\, A_1 \ldots A_n$ is a formula as well.[1] Special cases of this rule are:

[1] Please note that Wittgenstein never uses meta-symbols like "i_j" and "A_i" in the *Tractatus*. He uses, e.g., *p* to indicate *elementary* sentences as well as *complex* sentences. The context makes it clear which readings we get. Cp. section 1.4. Furthermore, if n = 0 in φ_i^n we would get the sentence without the operator: "(An elementary proposition is a truth-function of itself.)" (T5).

If A is formula, then $\sim A$ is formula. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \equiv B)$ and $(A|B)$ are formulas.[2] The rule (R3) is a conditioned as well as a recursive rule. We can take the outputs of this rule as new inputs.

Composition now means that we apply these rules forwards (→). Decomposition means that we apply these rules backwards (←) in an appropriate way. The point is that we have to differentiate between two types of composition and decomposition: case (a) the R1/R2/R2*-type and case (b) the R3-type. (b)-decomposition of a well-formed formula is possible because the logical form of this formula and each subformula has to show exactly one main operator and the logical form of the arguments. Here are some examples of composition and decomposition:

composition			
(1)	p	R1 →	(a)
(2)	Rab	R2 →	(a)
(3)	Rcd	R2 →	(a)
(4)	$\sim Rab$	R3 → (2)	(b)
(5)	$\sim\sim Rab$	R3 → (4)	(b)
(6)	$(p \vee Rcd)$	R3 → (1),(3)	(b)
(7)	$(\sim Rab \wedge Rcd)$	R3 → (4),(3)	(b)

decomposition (b)				
(1)	$(\sim(Rab \vee Rcd) \supset \sim Rab)$			
(2)	$\sim(Rab \vee Rcd)$			R3←(1)
(3)			$\sim Rab$	R3←(1)
(4)			Rab	R3←(3)
(5)	$(Rab \vee Rcd)$			R3←(2)
(6)	Rab			R3←(5)
(7)		Rcd		R3←(5)

decomposition (a)								
(8)	$[R]$	R2 (6)	(11)	$[R]$	R2 (7)	(14)	[R]	R2 (4)
(9)	a	R2 (6)	(12)	c	R2 (7)	(15)	a	R2 (4)
(10)	b	R2 (6)	(13)	d	R2 (7)	(16)	b	R2 (4)

(a)-Composition has to create the atomic / elementary formulas as inputs first. (b)-Composition can take any formerly created formulas to create new formulas. (b)-Decomposition stops at atomic / elementary formulas. (a)-Decomposition finally stops at the configuration sign "R" (if any) and at the terms "a", "b", "c" and "d". A complex expression is a formula iff this expression can be composed following the formation rules forwards only *and* can be decomposed following the formation rules backwards only. It is important that we have to differentiate between two types of atoms: The atoms / elementary formulas we get finally using only (b)-decomposition and the atoms if we apply (a)-decomposition. (a)-decomposition yields the "real" atoms because at this stage there is no further formation rule available which we could use backwards to get subparts of terms.

1.1 Logical Atomism 1 (LA1): Facts (Tatsachen) / Atomic Facts

In the Tractatus Wittgenstein makes a clear distinction between composition (a) and (b) as well as between decomposition (a) and (b). "The world is the totality

[2] Other notational versions of our classical conjunction "$(A \wedge B)$" are "$(A \,.\, B)$" and "$(A \,\&\, B)$".

of facts, not of things." (T1.1) A *fact* (Tatsache) is "created" by (a)-composition but the world is not created by (a)-composition. In this case the world could be described as the totality of things. That is wrong! But the world is also not the result of composition (b). From a logical point of view there is no connection between the facts in the world. The operators of the form φ_i^n – which are the basis of (b)-(de-)composition are not part of the world. In "The world divides into facts" (T1.2) "divides into" ("zerfällt in") does not indicate decomposition (b). There are no (b)-complexes in the world (cp. T4.0312). Especially, there are no negative facts in the world which form would be "$\sim p$". Negation is not an object!

For describing the "result" of (a)-composition Wittgenstein uses the expression "configuration" ("Konfiguration"): "The configuration of the objects forms the atomic fact." (T2.0272) "The substance of the world *can* only determine a form and not any material properties. For these are first presented by the propositions – first formed by the configuration of the objects." (T2.0231) Objects are "true" atoms. That is their logical status. "Objects form the substance of the world. Therefore they cannot be compound." (T2.021) Objects are free of any kind of composition – in the (b)- as well in the (a)-case. The configuration of facts could be seen as the "result" of using (a)-composition. An (elementary) fact is an (a)-composition with respect to (R2) or (R2*). We can take (elementary) facts as pictures. The possibility of a fact to be a *pictures* bases exclusively on its (a)-composition.

1.2 Logical Atomism 2 (LA2): Elementary Propositions (Elementarsätze)

In T 4.21 Wittgenstein introduces the concept "elementary proposition" ("Elementarsatz"): "The simplest proposition, the elementary proposition, asserts the existence of an atomic fact." The expression "the simplest proposition" indicates that if there are any subparts of this proposition they are not propositions. If there are subparts they are of another logical type. Elementary propositions are conditioned, non-recursively (a)-composed. Names are the only subparts of elementary propositions. Names are not propositions. Names – like objects – are the substance of any elementary proposition. Wittgenstein calls the (a)-composition of names "connexion" ("Zusammenhang") or "concatenation" ("Verkettung"): "The elementary proposition consists of names. It is a connexion, a concatenation, of names." (T4.22) The concatenation of names is not a class of names. We would get a class of names if we take only the names of an elementary poroposition and neglect its (a)-composition: "Only facts can express a sense, a class of names cannot." (T3.142) A concatenation of names is an (a)-composed fact which expresses – as a picture – a sense. It can be true/false.

Our (b)-composition do not play any role here. I. e., that our operators of the form φ_i^n (truth-functions) cannot be part of an elementary proposition. "$\sim p$" shows

– simply because of the occurrence of "~" (a special case of φ_i^1) – that this expression does not represent an elementary form. If we look at propositions which can be of the simplest form (elementary) or of a more or less complex form then Wittgenstein uses the term „articulate" („artikuliert"): „The proposition is articulate." (T3.141)

1.3 Logical Atomism 3 (LA3): Objects (Gegenstände) and Names (Namen) as Real Atoms

Objects as well as names are completely free of composition and, therefore, not decomposable. They are the real atomic parts of the (a)-composed facts or elementary propositions, respectively. Names are primitive signs (cp. T3.26). Names are in a sense also parts of complex propositions, but they are not direct input-data for (b)-compositions. Names are parts of a complex construction only as parts of (a)-compositions which can be used as input-data of (b)-compositions. Wittgenstein uses for expressions which are completely free of composition symbols which consist of exactly one sign: *"x", "y"* (variables), "*a*", "*b* " (constants) etc.

1.4 Logical Atomism and Holism 1 (LA4): Substitution Rule

Wittgenstein does not use different symbols for elementary and complex propositions. T4.24: "The elementary proposition I write as function of the names, in the form "*f x*", "*ϕ"(x,y)*", etc. / Or I indicate it by the letters *p*, *q*, *r*." In this case, e.g., "*p*" is a kind of shortening of any (a)-composed expression. In other cases "*p*" and "*q*" are used to represent complex propositions: "The truth-grounds of *q* are contained in those of *p*; *p* follows from *q*" (T5.121) Nowadays we could write the last part in the form"$q \models p$". In this case at least one of the expressions "*p*"/ "*q*" has to be a complex one (e.g., $r \models (r \vee s)$. Otherwise it would be impossible to get this inference or the case (TTTT)*pq*. Elementary propositions are logically independent of one other. In the same way in which atomic facts are independent of one other (cp. T2.061) it holds for all elementary propositions that an elementary proposition does not follow from another elementary proposition. (cp. T5.134)

But there is a reason why Wittgenstein does not use different symbols for (a)-composed expressions ("*p*") and (b)-composed expressions ("*A*", e.g. "$(r \vee s)$"). In an appropriate logical space we have a rule that says that there is no logical difference with respect to logical validity of a (b)-composed formula *A* ($\models A$) if we switch from any (a)-composed elementary proposition of the form p_i – with respect to all occurrences of p_i in A – to an arbitrary formula B. "$A\{p_i / B\}$" is read as "the result of substituting an arbitrary formula B for p_i in all occurrences of p_i in A". *The substitution rule* states: If $\models A$, then $\models A\{p_i / B\}$.

Given this rule we can characterize any expression of the form p_i elementary if and only if the substitution rule holds with respect to that expression without any restiction.

In the *Tractatus* as well as in *RLF* Wittgenstein does not give overt arguments in this direction. Later argumentation – e.g. with respect to *his non-negame* in appendix I of *Remarks on the Foundation of Mathematic* (RFM 102-110) – shows that he is aware of the possibility of giving up the (unrestricted) substitution rule.

2 New aspects of logical atomism in Wittgenstein's *Some Remarks on Logical Form* (1929)

In 1929 Wittgenstein got some new insights which forced him to come back to several aspects of logical atomism. Our impression is twofold: (i) He tries to keep his Tractarian program and at the same time (ii) he is criticizing several aspects which seem to be fundamental for keeping his program. Let us have a closer look!

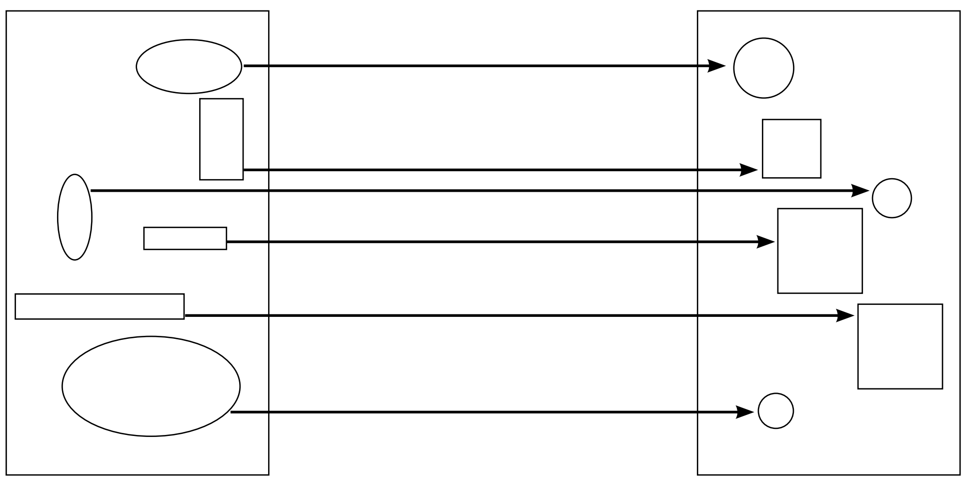
The first passage (p. 162) is about syntax in a very general sense. The rules of syntax for constants and variables should be the same. These rules "tell us in which connections only a word gives sense, thus excluding nonsensical structures" (RLF 162). What is striking here is his emphasis on the differences between logical syntax he is looking for and the "syntax of ordinary language". The last type "does not in all cases prevent the construction of nonsensical pseudopropositions" (Ib.).

Wittgenstein keeps the clear distinction between (a)- and (b)-(de-)composition. If we analyze ((b)-decompose) an arbitrary complex proposition – a proposition which contains "logical sums, products or other truthfunctions of simpler propositions" (Ib.) – we reach a final point: propositions which are not (b)-composed. (a)-composition is now called "the ultimate connection of the terms", "the immediate connection which cannot be broken without destroying the propositional form as such" (RLF 162 f.). It is astonishing that Wittgenstein calls "this ultimate connexion of terms" after Russell "atomic propositions" and not "elementary propositions" ("Elementarsätze") like in the *Tractatus*. This is surprising because he is willing to put more structure (numbers) into the (a)-composed expressions.

Like in the *Tractatus* the atomic propositions as "the kernels of every proposition" (RLF 163) contain all the material. (b)-composition "is only a development of this material" (Ib.). With respect to LA1 there is no significant difference.

Wittgenstein is still looking for "an appropriate symbolism" (RLF 163) "which gives a clear picture of the logical structure, excludes pseudopropositions, and uses its terms unambiguously". I.e., that the symbolism has to be *clear* /

precise. The method to get this symbolism is "by inspecting the phenomena which we want to describe, thus trying to understand their logical multiplicity" (Ib.). We get our correct analysis by means of "the logical investigation of the phenomena themselves" (Ib.). Wittgenstein feels that in the *Tractatus* he gives too much prominence to the idea that there are some designated forms of atomic propositions: the subject-predicate-form and relational forms suggested by the corresponding forms in ordinary language – logically explicated using elementary formulas with a function-argument-structure. The new enterprise is "in a certain sense *a posteriori*" because an "atomic form cannot be foreseen" (ib.) To show that we have to look for other (a)-composition rules he uses a simile which we represent by the following picture:



plane I: parallel to plane II:

ellipses of different sizes and shapes	circles (of different sizes and shapes)
rectangles of different sizes and shapes	squares (of different sizes and shapes)

The task is to produce images of the figures of plane I on plane II. One "possible convenient way of representation" is to "lay down the rule that every ellipse on plane I is to appear as a circle in plane II, and every rectangle as a square in II." (RLF 164) If we have only the picture of plane II and we know the indicated rule that a circle / a square on plane II goes back to an ellipsis / a rectangle on plane I, then we cannot immediately infer "the exact shapes of the original figures on plane I".[3] This case is analogous to the case concerning ordinary language: "If the facts of reality are the ellipses and rectangles on plane I the subject-predicate and relational forms correspond to the circles and squares in plane II. These forms are the norms of our particular language into which we project in ever so many different ways ever so many different logical forms. And

[3] The same holds for the exact sizes of the ellipses and rectangles. "In order to get in a single instance at the determinate shape of the original we would have to know the individual method by which, e.g., a particular ellipse is projected into the circle before me." (RLF 164)

for this very reason we can draw no conclusions—except very vague ones—from the use of these norms as to the actual logical form of the phenomena described." (RLF 164 f.)

With respect to LA2 there is a remarkable continuity: Elementary propositions cannot contain truth-functions like logical sums and logical products (cp. RLF 162). But there is no hint that Tractarian elementary propositions can contain *numbers*.[4] This is something that Wittgenstein demands directly: "And here I wish to make my first definite remark on the logical analysis of actual phenomena: it is this, that for their representation numbers (rational and irrational) must enter into the structure of the atomic propositions themselves." (RLF 165) The formalism must have the same logical multiplicity like the phenomena which it is a picture of. There are a lot of phenomena which indicate a kind of multiplicity which we cannot show without using numbers. "We meet with the forms of space and time with the whole manifold of spacial and temporal objects, as colours, sounds, etc., etc., with their gradations, continuous transitions, and combinations in various proportions, all of which we cannot seize by our ordinary means of expression" (RLF 165). This makes it clear that Wittgenstein is looking for atomic propositions with a significantly greater multiplicity than the (a)-composed first-order expressions – created by R2 or R2* – are able to provide.

In the *Tractatus* the true atomic subparts of atomic propositions are names represented usually by *one single symbol*. Now numbers have to take the position of names if they "must enter into the structure of the atomic propositions themselves". If we try to represent intervals by pairs of (connected) numbers, then we get much more complex symbols at the positions of names. This leads immediately to some modifications of LA3. Let us have a look at Wittgenstein's own examples.

If we try to represent a patch P we can use the notation "[6–9, 3–8]". In order to do this we have to assume a kind of coordinate system. But that means to assume a specific structure of the logical space. This is not in accordance with the Tractatarian position. From the old holistic point of view any assumption with respect to a special structure of the logical space seems to be misleading: "A speck in a visual field need not be red, but it must have a colour; it has, so to speak, a colour space round it. A tone must have a pitch, the object of the sense of touch a hardness, etc." (T2.0131) This sounds good. But: "Every picture is *also* a logical picture. (On the other hand, for example, not every picture is spatial.)" (T2.182) We can continue: "Not every picture has a colour". This means that logical space (space in general) has in an important sense no coordinate system. It should not exclude any specific kind of space (colour space, spatial space etc.). The occurrence of numbers in atomic propositions can be associated with some sort of coordinate systems, scales etc. as characterizing logical space. This gives us a kind of *a posteriori* understanding of logical space. "The system of co-ordinates here is part

[4] There is one exception: the number of arguments/names. But this number is *shown* and not used.

of the mode of expression; it is part of the method of projection by which the reality is projected into our symbolism." (RLF 165).

Now we are able to present new forms of atomic propositions: "The patch is red" can be represented as "[6–9, 3–8] R". If "The patch is red" is of subject-predicate-form "Is-Red(the patch)" represented by a function-argument-structure "R([6–9, 3–8])" the expression "[6–9, 3–8]" looks like a very complex name. Wittgenstein tells us that "'R'" is yet an unanalyzed term ('6–9' and '3–8' stand for the continuous interval between the respective numbers)" (RLF 166). Wittgenstein is on the way and not completely happy with his proposal: The proposal is not complete, e.g., because it does not mention time. (cp. ib.)

But Wittgenstein feels certain with respect to numbers: "The occurrence of numbers in the forms of atomic propositions is, in my opinion, not merely a feature of a special symbolism, but an essential and, consequently, unavoidable feature of the representation. And numbers will have to enter these forms when—as we should say in ordinary language—we are dealing with properties which admit of gradation, i.e., properties as the length of an interval, the pitch of a tone, the brightness or redness of a shade of colour, etc. It is a characteristic of these properties that one degree of them excludes any other." (RLF 166 f.) Wittgenstein mentions here a new kind of oppositions between atomic propositions (*exclusion*). If we allow new patterns of atomic propositions we can lose the logical independence between arbitrary atomic propositions. That means it is possible to formulate new rules of (a)-composition which allow the creation of two atomic propositions E1 and E2 with E1 excludes E2: "E1 & E2" yields "some sort of contradiction" (RLF 168).

With respect to "a proposition which asserts the existence of a colour R at a certain time T in a certain place P of our visual field" (ib.) Wittgenstein offers another form of (atomic) propositions: "R P T" (= our E1). It is remarkable that Wittgenstein avoids any hint to read this construction as an elementary first-order form. Maybe the place-time is red, the red is at some place-time, the place is T-red etc. Take "B P T" as our E2. Then it should be clear that "R P T & B P T" is "some sort of contradiction", "not merely a false proposition" (RLF 168), a "mutual exclusion" (RLF 168 f.). Wittgenstein says that the reason for this is, that constructions of the form "( ) P T"[5] "leaves room only for one entity—in the same sense, in fact, in which we say that there is room for one person only in a chair" (RLF 169).

Wittgenstein's argumentation is that there is a significant difference between "contradiction" and "mutual exclusion" ("some sort of contradiction"). "R P T & ~R P T" is really a contradiction:

R P T	~ R P T	R P T & ~ R P T
T	F	F
F	T	F

[5] Surprisingly Wittgenstein says "the function '( ) P T'" (RLF 169).

Here the second argument of the logical product is the negation of the first argument. If the arguments are independent of one another we get 4 distinct truth-value-assignments:

p	q	p & q
T	T	T
T	F	F
F	T	F
F	F	F

Both cases are distinct from the case of "R P T & B P T". Here the assumption is that the combination "T T" is not possible and we get only 3 distinct assignments:

R P T	B P T
T	F
F	T
F	F

The question is now how we can avoid the fourth assignment "T T". One idea would be to restrict our rule of (a)-composition contextually. If we have already created the formula "R P T" we are not allowed to create the formula "B P T" and vice versa. That is a bad idea because we lose too much multiplicity. We are not able to describe the possible case "R P T" = F and "B P T" = T. Wittgenstein's idea is: "the top line "T T T" must disappear, as it represents an impossible combination" (RLF 170). But how? Wittgenstein's general answer is: "a perfect notation will have to exclude such structures by definite rules of syntax" (RLF 171). But by what kind of rules of syntax?

3 Truth-function Within Atomic Propositions?

One of Wittgenstein's assumptions in *RLF* which is in accordance with the *Tractatus* is that operations (truth-functions) can not be used within atomic propositions. I will sketch two-dimensional systems which contain 2-dimensional expressions and operations which can be characterized by special reduction rules. These reduction rules allow a complete reduction of every formula to an 2-dimensional expression of the form $\begin{bmatrix} A \\ B \end{bmatrix}$. 2-dimensional validity ($\Vert\!\!=$) can be defined using only classical validity ($\Vert\!\!=$). Biunique mappings between the language of classical logic and languages of our two-dimensional logics will be given to show that these logical systems are equivalent. The interesting point is that atomic propositions of the form p_i can be translated into any ordered pair of classical formulas of the form. $\begin{bmatrix} A_i^1 \\ B_i^2 \end{bmatrix}$ What happens if we translate atomic propositions of classical logic into different types of ordered pairs of classical formulas and combine the resulting systems? We gain a nice-looking reconstruction of "R P T &

B P T". But then we can show that our ordered pairs of classical formulas acting as basic expression are not atomic in the sense of LA4. I.e., the rule of substitution fails (cp. section 1.4).

Let us begin with our language of classical logic = $\{p_i, \sim, \&, \vee\}$. We construct a 2-dimensional language = $\{\begin{bmatrix}A^1\\A^2\end{bmatrix}, -, \otimes, \oplus\}$. $\begin{bmatrix}A^1\\A^2\end{bmatrix}$ is the form of ordered pairs of classical formulas A^1 and A^2. A^1 and A^2 can be the same formula. – is a unary reduction operator. $\otimes$ and $\oplus$ are binary reduction operators. Formulas are of the forms $\begin{bmatrix}A^1\\A^2\end{bmatrix}$, $-X$, $(X \otimes Y)$, $(X \oplus Y)$, with X and Y containing no classical operators outside the scope of brackets.[6] Each reduction operator is characterized by a unique reduction rule.

$$-\begin{bmatrix}A^1\\A^2\end{bmatrix} \Rightarrow \begin{bmatrix}\sim A^1\\\sim A^2\end{bmatrix} \qquad \begin{bmatrix}A^1\\A^2\end{bmatrix} \otimes \begin{bmatrix}B^1\\B^2\end{bmatrix} \Rightarrow \begin{bmatrix}A^1 \wedge B^1\\A^2 \wedge B^2\end{bmatrix} \qquad \begin{bmatrix}A^1\\A^2\end{bmatrix} \oplus \begin{bmatrix}B^1\\B^2\end{bmatrix} \Rightarrow \begin{bmatrix}A^1 \vee B^1\\A^2 \vee B^2\end{bmatrix}$$

The reduction rule is a substitution device. If the left expression occurs in a formula it can be substituted by the right expression. A complete reduction yields a formula of the Form $\begin{bmatrix}A^1\\A^2\end{bmatrix}$. We define a new notion of validity: $\Vdash \begin{bmatrix}A^1\\A^2\end{bmatrix}$ iff $\vDash (A^1 \wedge A^2)$. Any 2D-formula X is valid ($\Vdash X$) iff its complete reduction $\begin{bmatrix}A^1_X\\A^2_X\end{bmatrix}$ (after using all possible reduction rules from inside to outside) is valid in the same sense: $\Vdash \begin{bmatrix}A^1_X\\A^2_X\end{bmatrix}$, i.e. $\vDash (A^1_X \wedge A^2_X)$.

We can choose any ordered pair of contingent classical formulas with the same index „i"! "Contingent" means that A^1_i as well as A^2_i are neither classical tautologies nor classical contradictions. Examples are $\begin{bmatrix}p_i\\p_i\end{bmatrix}$, $\begin{bmatrix}p^1_i\\p^2_i\end{bmatrix}$, but also $\begin{bmatrix}p_i\\\sim p_i\end{bmatrix}$ and $\begin{bmatrix}p^1_i \wedge \sim p^2_i\\\sim p^1_i \wedge p^2_i\end{bmatrix}$. I.e., that each of these 2D-expressions can play the role of an elementary proposition. To get a 2-dimensional system which is equivalent to classical logic we have to decide which form of such an ordered pair we would like to choose. We have to fix the structure which corresponds to atomic propositions of the form p_i. It can be a concrete structure but also a form: propositional variables $p_i \Leftrightarrow \begin{bmatrix}A^1_i\\A^2_i\end{bmatrix}$. The other biunique mappings are $\sim \Leftrightarrow -$ $\quad \wedge / \& \Leftrightarrow \otimes$ $\quad \vee \Leftrightarrow \oplus$.

[6] For reasons of space I am – unfortunately – very short here.

Classical operators (truth-functions) correspond to our reduction operators which are characterized by the reduction rules above.

Example 1: $p_i \Leftrightarrow \begin{bmatrix} p \\ p \end{bmatrix}$ $\qquad \vDash \sim(p \,\&\sim p)$ $\qquad$ translation: $\Vdash -\left(\begin{bmatrix} p \\ p \end{bmatrix} \oplus -\begin{bmatrix} p \\ p \end{bmatrix}\right)$

computation: $-\left(\begin{bmatrix} p \\ p \end{bmatrix} \oplus \begin{bmatrix} \sim p \\ \sim p \end{bmatrix}\right)$; $\qquad -\begin{bmatrix} p \wedge \sim p \\ p \wedge \sim p \end{bmatrix}$; $\qquad \Vdash \begin{bmatrix} \sim(p \wedge \sim p) \\ \sim(p \wedge \sim p) \end{bmatrix}$

because of $\vDash \sim(p \,\&\sim p)$ and $\vDash (\sim(p \,\&\sim p) \wedge \sim(p \,\&\sim p))$

Example 2: $p_i \Leftrightarrow \begin{bmatrix} p^1 \wedge \sim p^2 \\ \sim p^1 \wedge p^2 \end{bmatrix} \vDash \sim(p \,\&\sim p)$ translation $\Vdash -\left(\begin{bmatrix} p^1 \wedge \sim p^2 \\ \sim p^1 \wedge p^2 \end{bmatrix} \oplus -\begin{bmatrix} p^1 \wedge \sim p^2 \\ \sim p^1 \wedge p^2 \end{bmatrix}\right)$

computation: $-\left(\begin{bmatrix} p^1 \wedge \sim p^2 \\ \sim p^1 \wedge p^2 \end{bmatrix} \oplus \begin{bmatrix} \sim(p^1 \wedge \sim p^2) \\ \sim(\sim p^1 \wedge p^2) \end{bmatrix}\right)$ $\quad -\begin{vmatrix} (p^1 \wedge \sim p^2) \wedge \sim(p^1 \wedge \sim p^2) \\ (\sim p^1 \wedge p^2) \wedge \sim(\sim p^1 \wedge p^2) \end{vmatrix}$

$$\Vdash \begin{vmatrix} \sim((p^1 \wedge \sim p^2) \wedge \sim(p^1 \wedge \sim p^2)) \\ \sim((\sim p^1 \wedge p^2) \wedge \sim(\sim p^1 \wedge p^2)) \end{vmatrix}$$

because of $\vDash \sim((p^1 \wedge \sim p^2) \wedge \sim(p^1 \wedge \sim p^2)) \wedge \sim((p^1 \wedge \sim p^2) \wedge \sim(p^1 \wedge \sim p^2))$ etc.

With respect to our biunique mappings between a classical formula A_Z and its analogue Z_A we get:

For all A_Z and Z_A: $\vDash A_Z$ iff $\Vdash Z_A$.

Let us use different biunique mappings concerning classical atomic propositions. The language is now $\{p_i, q_j, \sim, \&, \vee\}$. There is no difference here from the classical point of view.

$$p_i \Leftrightarrow \begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix} \text{ and } q_i \Leftrightarrow \begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}.$$

Let "p_i" be the representation of "R P T". "p_i^1" says what "p_i" *says* = "R P T". p_i^2 stands for "P T has another colour, e.g. blue": "B P T". $\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}$ *says* that "R P T & ~B P T" and *presupposes* the other possibility "~R P T & B P T". If we look at the assignments of truth-values we get the following picture

p_i^1	$\wedge$	$\sim$	p_i^2
T	**F**	F	T
T	**T**	T	F
F	**F**	F	T
F	**F**	T	F

$\sim$	p_i^1	$\wedge$	p_i^2
F	T	**F**	T
F	T	**F**	F
T	F	**T**	T
T	F	**F**	F

We observe that we get only the combinations "F F" (twice), "T F" and "F T". There is no combination "T T". "$p_i^1 \wedge \sim p_i^2$" and "$\sim p_i^1 \wedge p_i^2$" cannot be true together. It is possible that "q_i" says what "p_i^2" says: "B P T". I.e., we have the opposite situation: $\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$ says that "~R P T & B P T" and presupposes the other possibility

"R P T & ~B P T". If we put "p_i" and "q_i" together we do not get a contradiction. But $\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix} \otimes \begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$ yields in the first dimension

$(p_i^1$	$\wedge$	$\sim$	$p_i^2)$	$\wedge$	$(\sim$	p_i^1	$\wedge$	$p_i^2)$
T	**F**	F	T	**F**	F	T	**F**	T
T	**T**	T	F	**F**	F	T	**F**	F
F	**F**	F	T	**F**	T	F	**T**	T
F	**F**	T	F	**F**	T	F	**F**	F

We get the same result if we look at $(\sim p_i^1 \wedge p_i^2) \wedge (p_i^1 \wedge \sim p_i^2)$ in the second dimension. Again, we get all the final occurrences of „F" because there is no occurrence of the combination „T T". But we have several occurrences of the combination „F F". What happens if we take our translation of $p_i \vee q_i$? $\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix} \oplus \begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix} = \begin{bmatrix} p_i^1 \not\equiv p_i^2 \\ p_i^1 \not\equiv p_i^2 \end{bmatrix}$. "R P T $\vee$ B P T" cannot have an inclusive reading because there is no situation with "T T":

$(p_i^1$	$\wedge$	$\sim$	$p_i^2)$	$\vee$	$(\sim$	p_i^1	$\wedge$	$p_i^2)$
T	**F**	F	T	**F**	F	T	**F**	T
T	**T**	T	F	**T**	F	T	**F**	F
F	**F**	F	T	**T**	T	F	**T**	T
F	**F**	T	F	**F**	T	F	**F**	F

In each logic with the language $\{\begin{bmatrix} A_i^1 \\ A_i^2 \end{bmatrix}, -, \otimes, \oplus\}$ we get the rule of substitution of the following form: $\dfrac{\Vdash X}{\Vdash X\{\begin{bmatrix} A_i^1 \\ A_i^2 \end{bmatrix}/Y\}}$ with X and Y representing arbitrary formulas of our language. In a logic with the language $\{\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}, -, \otimes, \oplus\}$ the rule of substitution $\dfrac{\Vdash X}{\Vdash X\{\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}/Y\}}$ is fine. We get the same result with respect to $\{\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}, -, \otimes, \oplus\}$ and the rule $\dfrac{\Vdash X}{\Vdash X\{\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}/Y\}}$. I.e., in an appropriate logical environment $\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$ as well as $\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$ act like elementary propositions.

What happens if we combine these different types of elementary 2D-expressions in one and the same language? We cannot keep our substitution rule in an unrestricted version. The new language is: $\{\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}, \begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}, -, \otimes, \oplus\}$

together with both substitution rules. Let "X" be "$-\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix} \oplus -\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$". It is easy to check that this formula is $\Vdash$- valid. Let "Y" be "$\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$". "$X\{\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}/Y\}$" gives then "$-\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix} \oplus -\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$" which is of course not $\Vdash$- valid. I.e., that in a logic with the language $\{\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}, \begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}, -, \otimes, \oplus\}$ neither $\begin{bmatrix} p_i^1 \wedge \sim p_i^2 \\ \sim p_i^1 \wedge p_i^2 \end{bmatrix}$ nor $\begin{bmatrix} \sim p_i^1 \wedge p_i^2 \\ p_i^1 \wedge \sim p_i^2 \end{bmatrix}$ can act as an elementary proposition.

4 Exclusion of Atomic Propositions on Colours and its Analogy With Atomic Propositions Describing Positions of Pieces on a Chess Board

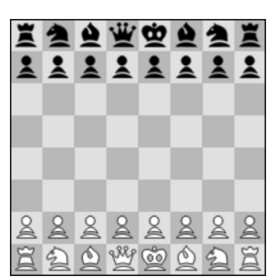
Let us investigate several forms of atomic propositions on colours by analogy with expressions to describe positions of (chess) pieces on a (chess) board, e.g., "the white queen is on the square xy (after the 17th move of the player who has the black pieces)". We can see then the differences between arbitrary positions of pieces on a board and positions of chess pieces on a chess board. The analogies rest on the following correspondences:

color words example: "R" ("red")	names of (chess) pieces examples: "Qw" ("white queen"), "Kw" ("white king), "Rb" ("black rook"), "Pw" (white pawn)
a patch P: example: "[6–9, 3–8]" (or "P" for place)	a square on the (chess) board example: "e4" (eth-file/4th rank)
time "T"	situation after the n^{th} move of the player who has the white/ black pieces – example: 17thb
form of an atomic proposition 1: example: "R[6–9, 3–8]"	form of a description of the position of a piece on a board which has nothing to do with the game chess: "Qw e4"
form of an atomic proposition 2: example: "R P T"	form of a description of the position of a *chess* piece on a *chess* board after the n^{th} move of white/black: example: "Qw e4 17thb" = "white queen on e4 after the 17th move of the player who has the black pieces"

"Qw e4" and "Qw e4 17thb" describe positions of pieces and not moves. The atomic proposition "Qw e4" is true if the white queen is on square e4. The atomic proposition " Qw e4 17thb" is true if the white queen is on square e4 after the 17th move of the player who has the black pieces. It does not matter when the white queen has reached e4 and how long she is already there. "Qw e4" has nothing to do with the chess *game*. The chess board is nothing else then a kind of finite coordinate system. In the codex of the FIDE-rules ("Laws of Chess") we find a neutral characterization of a board. It is "composed of an 8 x 8 grid of 64 equal squares alternately light (the 'white' squares) and dark (the 'black' squares)."

(Article 2.1) If the task would be to put some pieces on a board it can occur that someone put two pieces on the same square or one piece on two squares. For our task the physical fact that it is impossible to put two pieces on the same square (the squares are too small with respect to our pieces) does not matter. I.e., that "Qw e4 & Rw e4" is not necessarily an exclusion in one and the same situation on the board. " Qw e4 17^{th}b & Kw e4 35^{th}w" is of course not problematic. Within a chess match between 17^{th}b and 35^{th}w the player who has the white pieces could have moved her queen to another square or the white queen was captured and removed from the board. Furthermore, this player has moved her king to e4 – maybe capturing a black piece (except the king) and removing it from the board. But what about "Qw e4 35^{th}b & Kw e4 35^{th}b" and about "Qw e4 35^{th}w & Kw e4 35^{th}b". We know from the basic rules[7] of the game that during a chess match it is impossible to put two pieces on one and the same square in the same chess position (same match) on a chess board. That's why "Qw e4 35^{th}b & Kw e4 35^{th}b" is "some sort of inconsistency". "Qw e4 35^{th}b" and "Kw e4 35^{th}b" mutually exclude one another. Both sentences can be true but not within the same match. Distinct parameters ("35^{th}w", "35^{th}b") with respect to the moving history do not help. We know with respect to the basic rules of chess that there is no possible chess history to put the white queen away from e4 and move the white king to e4 using only one black move. There are isolated sentences which describe positions of a chess piece which are impossible with respect to some basic rules of chess: "Pw e1". In such a case "Pw e1 35^{th}w " is also impossible. And there are descriptions of piece positions which are impossible with respect to the basic rules including the initial position: "Qw e4 2^{nd}w" – it takes at least 3 white moves to reach e4 (with or without the help of the player who has the black pieces).

Except pawns and bishops all the other chess pieces can theoretically reach any square of the board. Our syntax needs a sufficient multiplicity to describe any of these positions. All propositions mentioned above should be expressible in this syntax. But how can complex sentences like "Qw e4 35^{th}b & Kw e4 35^{th}b" be blocked? Where is no need to use this kind of propositions because there is nothing to describe during any chess match due to the basic rules. In paragraph 2.3 of the "Laws of Chess" (FIDE) the initial position of the pieces on a chessboard is given by a picture:



[7] Among the basic rules of chess are at least the rules characterized in article 2 "The initial position of the pieces on the chessboard" and article 3 "The moves on the pieces".

We see the light ('white) and dark ('black') squares. We observe that there are squares occupied by chess pieces and that there are empty squares. In this position each square is occupied by at most one piece. Each piece occurs exactly on one square and not (partly) on two or four squares. In paragraph 3.1 we read: "It is not permitted to move a piece to a square occupied by a piece of the same colour. If a piece moves to a square occupied by an opponent's piece the latter is captured and removed from the chessboard as part of the same move." So we learn that it is impossible to find any sequence of moves that leads to a position in which two pieces occupy the same square. We get two different understandings of (chess) boards as coordinate systems: (a) A board is nothing else than a geometrical structure ("an 8 x 8 grid of 64 equal squares alternatively light ... and dark"). This is enough to characterize the position of arbitrary objects on the board. This (logical) multiplicity is too high with respect to the basic rules of chess. It allows that several objects can occupy the same region / the same square. It allows that one object can (partly) occupy more than one square. (b) If it comes to a chess board we have to consider not only its geometry but also the basic rules of chess. The coordinate system is not independent from these rules. The coordinate system of chess is such that each square is occupied by at most one piece and each piece on the board occupies exactly one square. We assume that we use atomic expression like ""Qw e4 35thb" to describe chess pieces on a chess board. How should the syntax of atomic expressions should look like assuming that this syntax has the same multiplicity?

5 Wittgenstein's Perspective

What makes the analogy between chess and the logical form of our colour system problematic is that we do not have an analogue to our chess board at hand. The argumentation in the section before points towards Wittgenstein's investigations in his later period.

In *RLF* Wittgenstein keeps his *Tractarian* presupposition that there should be exactly one logic: "We must eventually reach the ultimate connection of the terms, the immediate connection which cannot be broken without destroying the propositional form as such." (RLF 162 f.) This position is still connected with logicism in the sense that we can find something which is logically fundamental: "Such rules, however, cannot be laid down until we have actually reached the ultimate analysis of the phenomena in question. This, as we all know, has not yet been achieved." (RLF 171) But putting numbers "into the structure of the atomic propositions themselves" and assuming any kind of coordinate system within logical space allow a kind of formal interaction between the structured atomic propositions and the coordinates of the logical space. One consequence is that some complex expressions are not allowed. Can we maintain logical atomism in the long run? Can we maintain at least some of its aspects?

Wittgenstein is on the way from a holistic understanding of logic to a holistic understanding of grammar. Shortly after RLF we have the following remark from

Waismann's notes for 25 December 1929: "I once wrote: 'A proposition is laid like a yardstick against reality. Only the outermost tips of the graduation marks touch the object to be measured.' I should now prefer to say: a *system of propositions* ['Satzsystem'] is laid like a yardstick against reality." (PR 317) Atomic propositions which contain numbers and the step from propositions to a *system* of proposition are cornerstones on Wittgenstein's journey to *language games*. This is combined with the understanding of logic as *grammar*, i.e., as an *open* structure opposed to an understanding of logic as a *calculus* which in some sense is a *closed* structure. We have to consider a lot of *language games*. Their interrelations should be characterized by (family) resemblances and not by structural (set-theoretical) interrelations between calculi.

References

WITTGENSTEIN, L.: Some Remarks on Logical Form [RLF]. *Proceedings of the Aristotelian Society, Supplementary Volumes, v. 9, Knowledge, Experience and Realism* (1929), p. 162-171.

_____ . *Tractatus Logico-Philosophicus* [T], Translated from the German by C. K. Odgen, with an Introduction by Bertrand Russell, London, Boston and Henley: Routledge & Kegan Paul 1981. Online: K. C. Klement: Side-by-side-by-side edition, version 0.29 (November 17, 2012), containing the original German, alongside both the Ogden/Ramsey, and Pears/McGuinness English translations., http://people.umass.edu/phil335-klement-2/tlp/tlp.html#dedication.

_____ . *Philosophical Remarks* [PR], edited by R. Rhees and translated by R. Hargreaves and R. White, Oxford: Basil Blackwell 1975.

WITTGENSTEIN, L. *Remarks on the Foundation of Mathematics* [RFM], edited by G. H. von Wright, R. Rhees, G. E. M. Anscombe, translated by G. E. M. Anscombe, Oxford: BasilBlackwell, 1994.

www.fide.com/FIDE/handbook/LawsOfChess.pdf: *Laws of Chess*. The English text is the authentic version of the Laws of Chess, which was adopted at the 79th FIDE Congress at Dresden (Germany), November 2008, coming into force on 1 July 2009.